EEEN203
Circuit Analysis
Laplace Warmup

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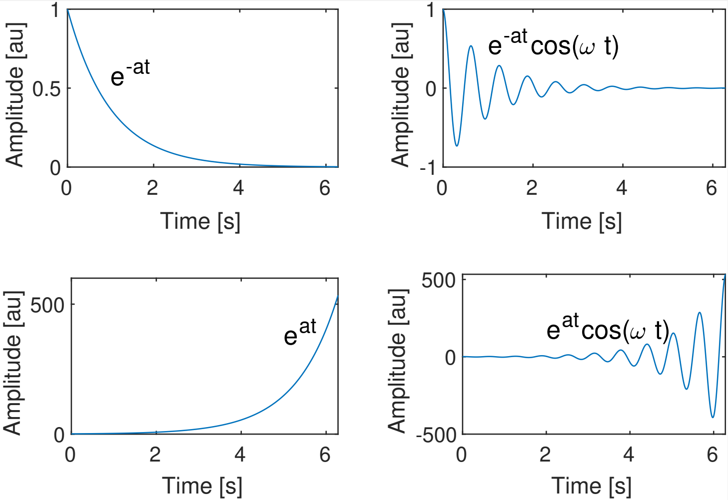
Revision 111

Quiz!

What sorts of reponses have you seen in circuits so far?

What about in mechanical systems?

Responses



Amplitude [au]
Time [s]
e-at
e-atcos(ω t)
eat
eatcos(ω t)
0
0.5
1
-1
500
-500
2
4
6

Responses

The response of *all* linear, constant-coefficient, ordinary differential equations is made up only of exponentials and exponentially modulated sinusoidal signals.

Such DEs look like

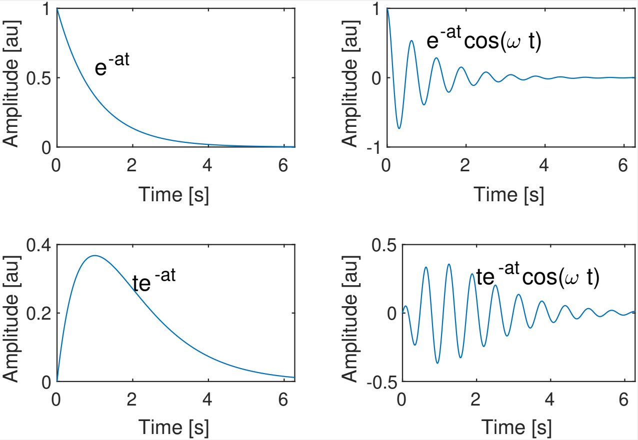
$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{LC} i = 0 \qquad \text{(Series Resonant circuit)}$$

$$J\ddot{\theta} + \left(b + \frac{K_\mathrm{t} K_\mathrm{e}}{R_\mathrm{a}}\right)\dot{\theta} = \frac{K_\mathrm{t}}{R_\mathrm{a}} v_\mathrm{a} \qquad \text{(DC motor)}$$

Here the coefficients of the derivatives are all real, unchanging constants, that come from the physical or electrical properties of components.

Most of the DEs we encounter in engineering work look like this, or can at least be approximated this way.

Special Case



The claim above about exponentials and exponentially modulated sinusoids is a bit strong. In special cases we will see we can also get those waveforms multiplied polynomials of time. We will return to this later.

The Laplace Transform

Linear, constant-coefficient ODEs can be efficiently and easily solved analytically using the Laplace transform. (Once you know the tricks.)

We will use it mostly for electrical and mechanical systems, but it can be used for any systems governed by these sorts of DEs (fluids, optics, heat, finances etc).

The Laplace transform will provide a language to talk abstractly about the properties of systems.

The Laplace Transform

The Laplace transform of a signal $f(t)$ is defined by the integral

$$\mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t$$

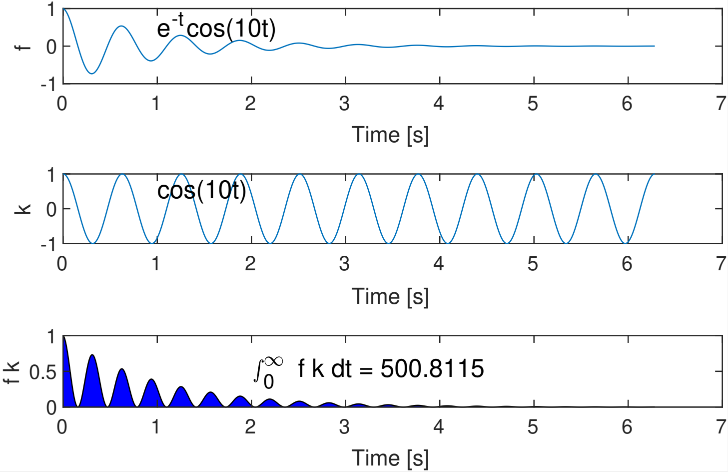
with s being a complex number, $s := \sigma + \mathrm{j}\omega$.

We normally write it in this convenient form, but this is equivalent to

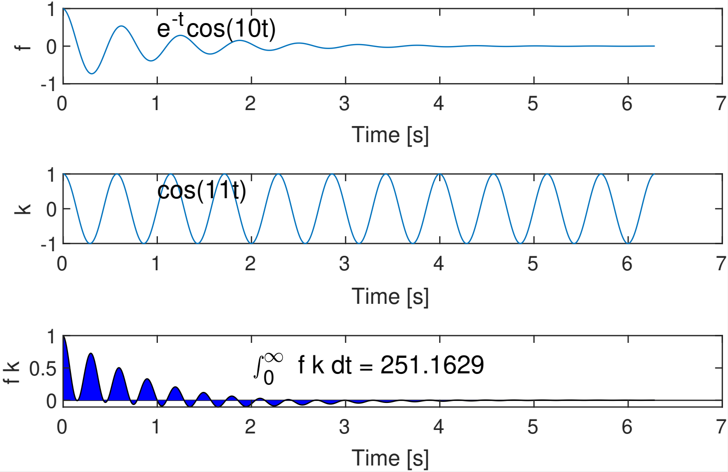
$$\mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} f(t)\mathrm{e}^{-\sigma t}\cos(\omega t)\,\mathrm{d}t$$

The Laplace transform can be infromally thought of as measuring how much our function $f(t)$ looks like an exponentially decaying sinusoid with particular values for σ and ω.

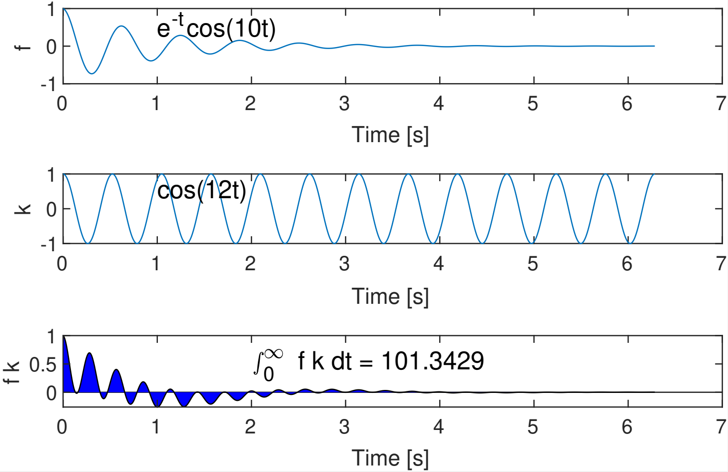
Laplace Integral



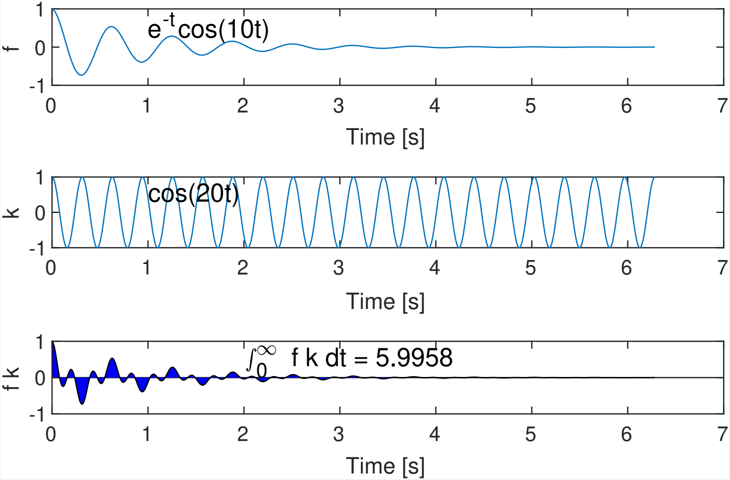
Laplace Integral



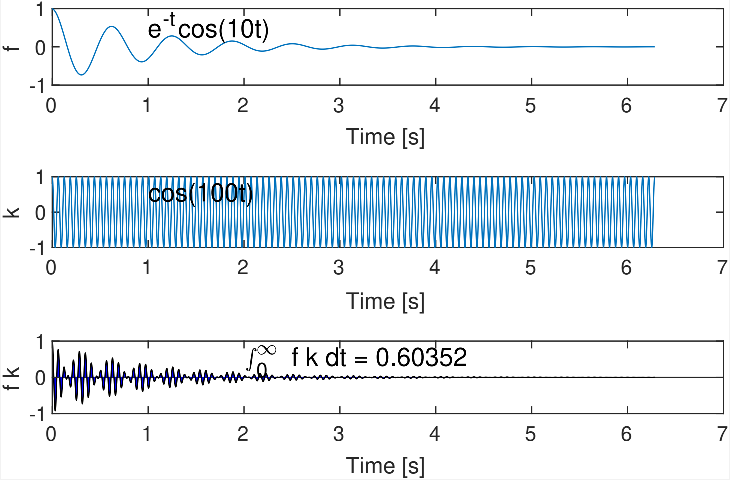
Laplace Integral



Laplace Integral



Laplace Integral



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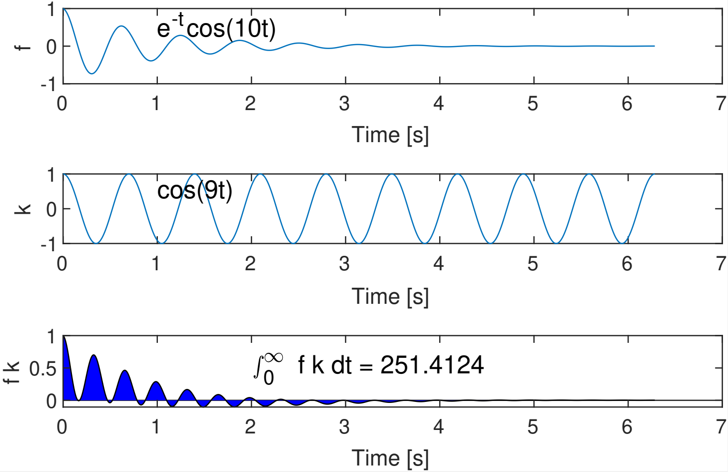

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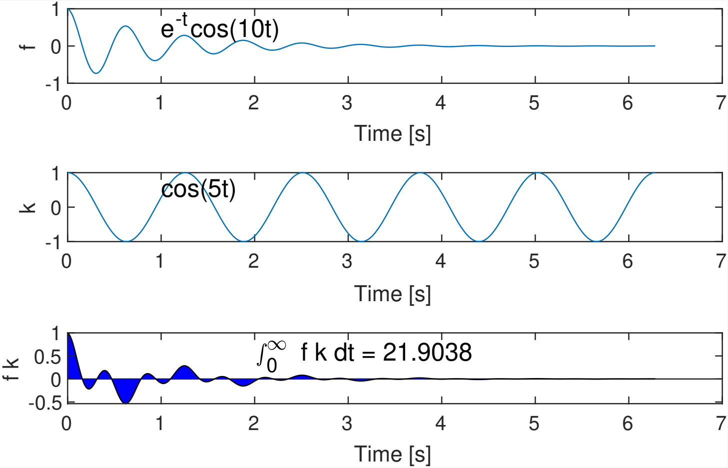

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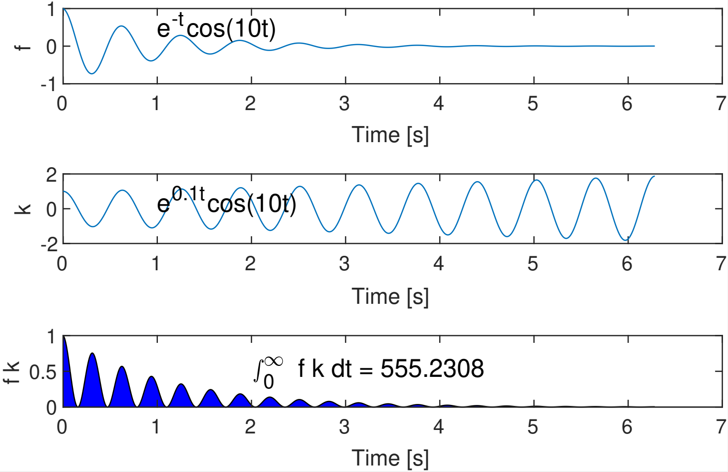
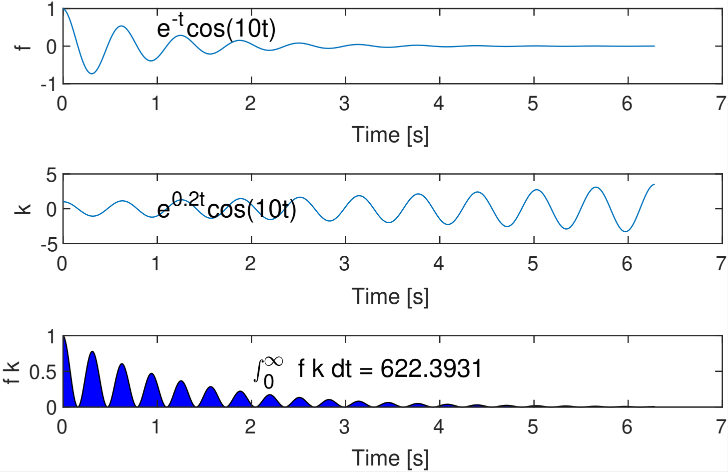


Laplace Integral



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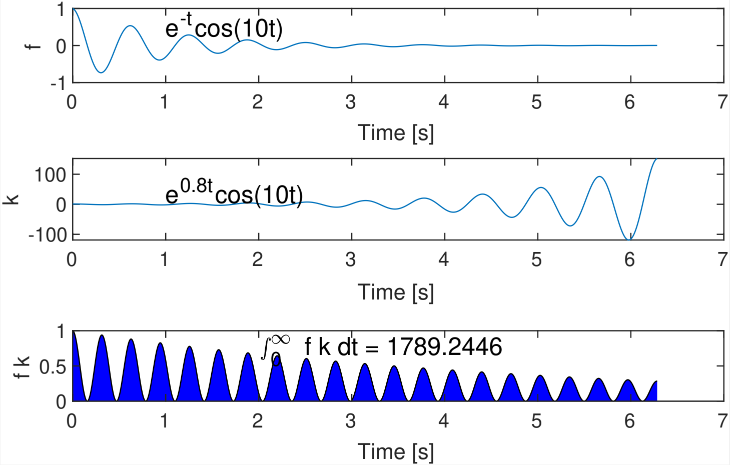

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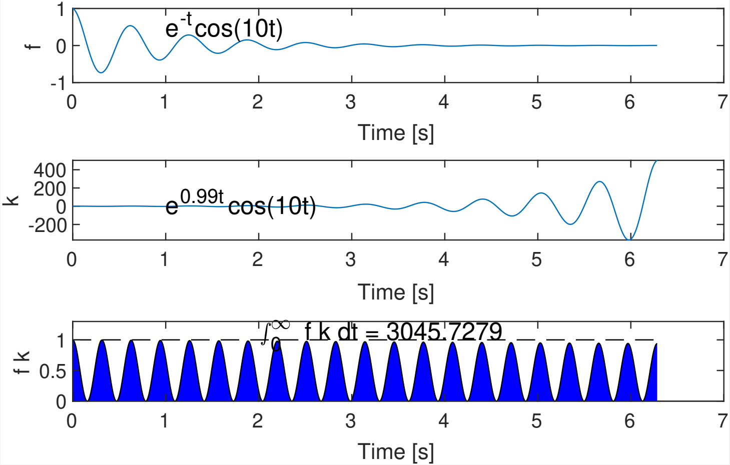
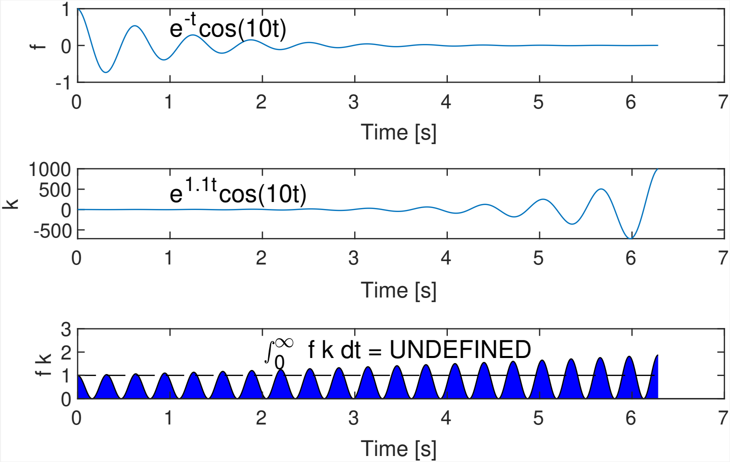
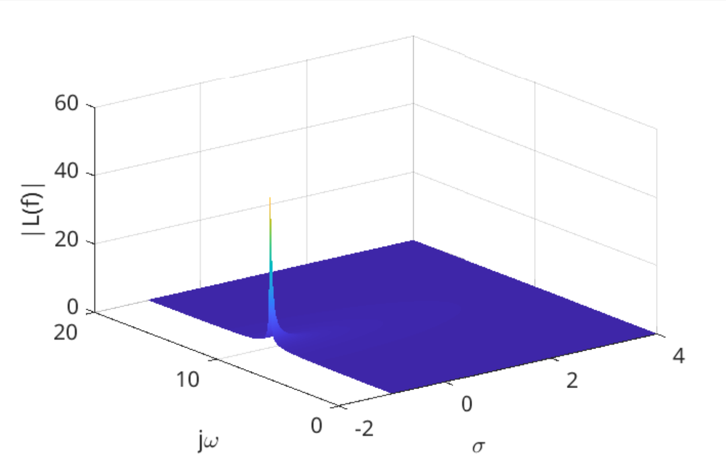


Laplace Integral



Magnitude of $\mathcal{L}\{e^{-t}\cos(10t)\}$



Notice that the response has a peak at $s = \sigma + j\omega = -1 + 10j$.

The area to the left of $\sigma = -1$ is outside the Region of Converegence (ROC) of the Laplace Transform.