EEEN203
Circuit Analysis
Systems Analysis using the Laplace Transform

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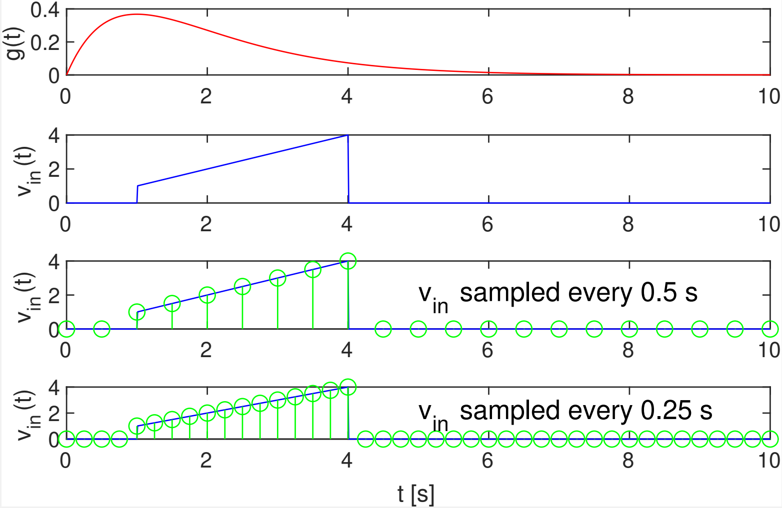
Revision 98

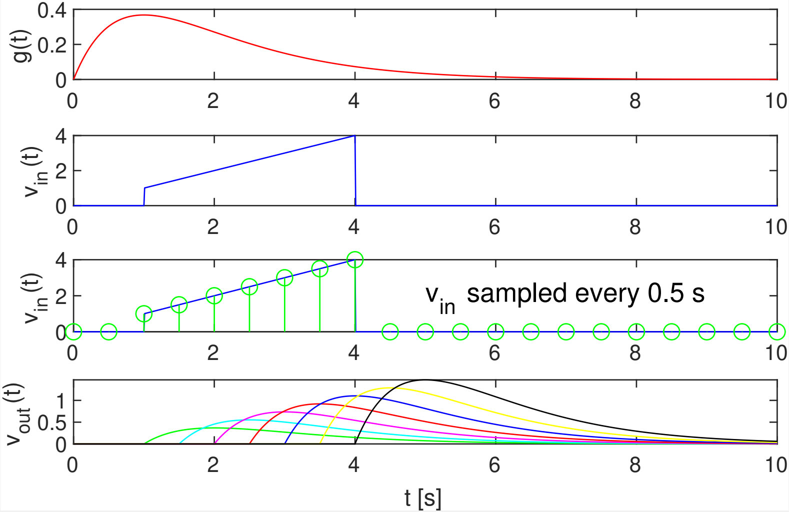
Impulse Response to General Response

If we know how a system responds to an impulse, then (in principle) we could work out how it responds to *any* input. We could divide the input signal into an infinitely fine set of “samples” and then let every sample set off an impulse response. Since the system is linear, we could find the overall system response by adding together all of these impulse responses.

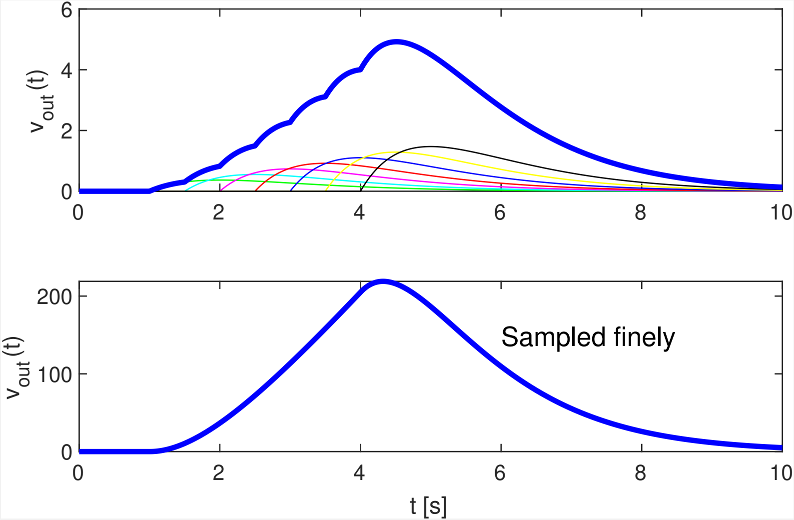
Sampling the input



Impulse Responses



Convolution



Convolution

The operation we have just performed, *Convolution*, occurs frequently and is one of the most important ideas in engineering, particularly electronic engineering. You will meet it again in many places.

It is perhaps most easily understood graphically, but we can define it mathematically as

$$f(t) * g(t) := \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \, \mathrm{d}\tau$$

Note that $*$ is the symbol for convolution, *not* multiplication! (That said, you will also somtimes see $\star$ used to denote convolution.)

Convolution and the Laplace Transform

Despite its importance, we try to avoid calculating convolution integrals. Instead, we rather use the Laplace (or Fourier) transforms.

We will not prove it here, but the Laplace transform turns convolution in the time domain into multiplication in the s-domain.

$$\boxed{\mathcal{L}\left\{f(t) * g(t)\right\} = \mathcal{L}\left\{f(t)\right\}\mathcal{L}\left\{g(t)\right\}}$$

or

$$\boxed{f(t) * g(t) \Longleftrightarrow F(s)G(s)}$$

We use the reverse property less often, but for completeness,

$$f(t)g(t) \Longleftrightarrow \tfrac{1}{2\pi \mathrm{j}} F(s) * G(s)$$

Convolution and the Laplace Transform

We have already seen this property in action several times, though not drawn attention to it. In particular, it is what makes transfer functions a useful idea.

$$V_{\text{out}}(s) = G(s)V_{\text{in}}(s)$$

is, in general, easier to deal with than

$$v_{\text{out}}(t) = g(t) * v_{\text{in}}(t)$$

Combining Systems

Transfer functions provide a convenient abstraction of the dynamics of a system. If we know the transfer function of a system we no longer need to worry about its internal mechanisms.

Transfer functions can be manipulated and combined as “objects” to build more complex systems. Typically we call these objects *blocks*, each of which has a transfer function.

Some questions arise:

- How do we combine transfer functions?
- What does this combination do to the modes (poles) of the systems?

Connecting blocks in series is very common; we do something to a signal and then take the output and perform a second action.

$$X(s) \longrightarrow \boxed{G_1(s)} \xrightarrow{Y(s)} \boxed{G_2(s)} \longrightarrow Z(s)$$

We can combine the transfer functions of the two blocks. By definition we have $\frac{Y(s)}{X(s)} = G_1(s)$ and $\frac{Z(s)}{Y(s)} = G_2(s)$. Therefore

$$\begin{aligned} Z(s) &= G_2(s)Y(s) \\ &= G_2(s)G_1(s)X(s) \\ \implies \frac{Z(s)}{X(s)} &= G_2(s)G_1(s) \end{aligned}$$

The transfer function of subsystems placed in series is the product of the subsystem transfer functions. This treatment can obviously be extended to greater numbers of successive stages.

Poles of blocks in series

We would like to know what the modes of a series system look like (or equivalently we want to know the pole locations). Let $G_1(s) = \frac{N_1(s)}{(s+a_1)(s+a_2)\cdots(s+a_n)}$ and $G_2(s) = \frac{N_2(s)}{(s+b_1)(s+b_2)\cdots(s+b_m)}$ for $a_i, b_i \in \mathbb{C}$ and $N_i(s)$ some polynomials.

The product of these yields

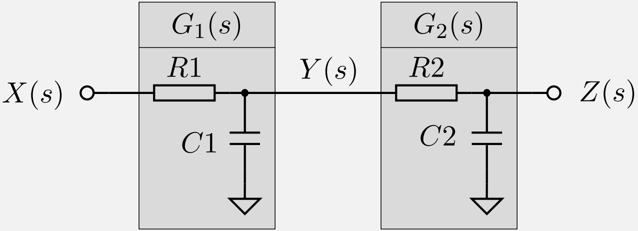
$$G(s) = \frac{N_1(s)N_2(s)}{(s+a_1)(s+a_2)\cdots(s+a_n)(s+b_1)(s+b_2)\cdots(s+b_m)},$$

so the poles of $G(s)$ are just the combination of the poles of its constituent parts. Placing two blocks in series does not alter the poles or the modes of the subsystems.

The numerator of the combined system is also the product of the subsystem numerators, so the zeros do not change either.

Loading

The previous analysis assumes that the transfer function of a block *does not change* when we connect another block to its output. We need to be careful in practice, because sometimes this is not true.



If $G_2(s)$ were not present, then $G_1(s) = \frac{Y(s)}{X(s)} = \frac{Z_{C1}}{R1+Z_{C1}}$. However, when we connect the second block having $G_2(s)$, then it *loads* $G_1(s)$. In this case, $G_1(s) = \frac{Y(s)}{X(s)} = \frac{Z_{C1} \| (R2+Z_{C2})}{R1+\left(Z_{C1} \| (R2+Z_{C2})\right)}$.

Buffering

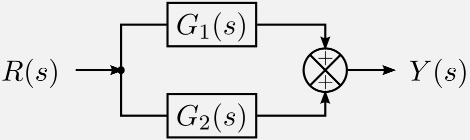
Often we do not have complete control over the subsystems with which we are working. For example, we might not be able to choose the input and output impedances of a circuit.

One method for overcoming the loading problem is to ensure that your system has buffering between the various subsystems. For example, it is very common to include op-amp buffers (amplifiers with unity gain) between stages to reduce their interaction.

Buffering often makes systems “work better”, but also makes it easier to design and analyse them.

Blocks in parallel

We also might want to combine blocks in parallel. That is, two blocks might have the same inputs and outputs.



In this case the transfer function is simply the sum of the transfer functions of the two (or more) subsystems.

$$\frac{Y(s)}{R(s)} = G_1(s) + G_2(s)$$

Poles and modes of blocks in parallel

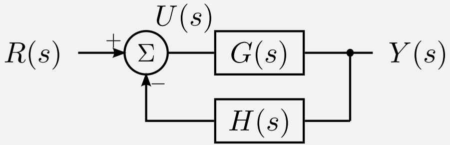
Let $G_1(s) = \dfrac{N_1(s)}{D_1(s)}$ and $G_2(s) = \dfrac{N_2(s)}{D_2(s)}$. Let's find an expression $G(s)$ for the transfer function of two combined in parallel.

$$\begin{aligned} G(s) &= \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{D_2(s)} \\ &= \frac{N_1(s)D_2(s) + N_2(s)D_1(s)}{D_1(s)D_2(s)} \end{aligned}$$

As in the case of series blocks, we again see that the denominator is the product of the subsystem denominators. Consequently a parallel combination will again show a combination of modes arising from the two subsystems.

The parallel combination leads to a more complicated numerator for the transfer function, which can produce different zeros and will typically change the amplitudes of the modes significantly.

In a feedback configuration we sample some of the output of the block with transfer function $G(s)$ and feed it back to the input.



$$
\begin{aligned}
Y(s) &= G(s)U(s) \\
&= G(s)\big[R(s) - H(s)Y(s)\big] \\
&= G(s)R(s) - G(s)H(s)Y(s) \\
\big[1 + G(s)H(s)\big]Y(s) &= G(s)R(s) \\
Y(s) &= \frac{G(s)}{1 + G(s)H(s)}R(s) \\
\implies T(s) := \frac{Y(s)}{R(s)} &= \frac{G(s)}{1 + G(s)H(s)}
\end{aligned}
$$

Poles in feedback systems

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

We call $T(s)$ the *closed loop* transfer function for the system, and $G(s)$ the open loop response.

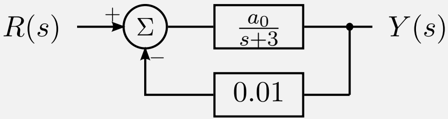
$T(s)$ is more interesting that the transfer functions produced by the other block combinations, because the poles of $T(s)$ need not be the same as those of $G(s)$ or $H(s)$. If we are given some system with transfer function $G(s)$ we often design an appropriate $H(s)$ to improve the pole location (and modes).

Feedback is one of the fundamental tools in the electronic engineering toolbox. We will see its power repeatedly over coming courses.

Feedback Example - The Operational Amplifier

A typical operational amplifier has a transfer function of something like $G(s) = \frac{a_0}{s+3}$ with $a_0 \approx 1 \times 10^6$. In practice we never use the amplifier "open loop", but always enclose it within a feedback network to change the properties of the overall system.

As an example, consider using a feedback block with transfer function $H(s) = 0.01$.



$$
\begin{aligned}
T(s) &= \frac{G(s)}{1 + G(s)H(s)} \\
&= \frac{\frac{a_0}{s+3}}{1 + \frac{0.01a_0}{s+3}} \\
&= \frac{a_0}{s + 3 + 0.01a_0}
\end{aligned}
$$

Feedback Example - The Operational Amplifier

Compare the open and closed loop transfer functions.

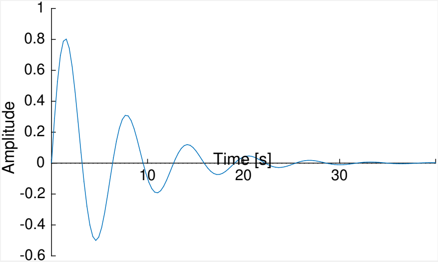
$$G(s) = \frac{a_0}{s+3} \qquad T(s) = \frac{a_0}{s+3+0.01a_0}$$

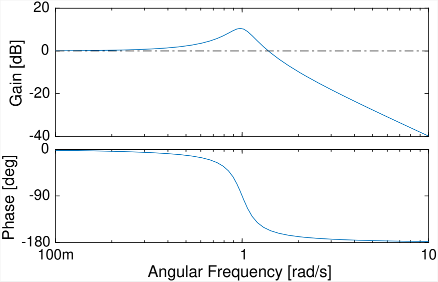
We have moved the pole from $s=-3$, to $s=-3-\frac{a_0}{100}$. As a_0 is large, this means that the mode is a lot faster.

Notice also that the gain is reduced. We can use the final value theorem to find the response to a step. This leads to a final value of a_0 for G, but only 100 for T.

Feedback Example - Resonant Circuit

Consider an RLC resonant circuit. For a particular example, let's consider a circuit that has a transfer function $G(s) = \dfrac{1}{s^2 + 0.3s + 1}$.
The resulting impulse and frequency responses of the circuit are shown in the figures.





Feedback Example - Resonant Circuit

The circuit "rings" quite a lot in the domain which would often be undesirable. We can also see that it would amplify noise or interference at around 1 rad/s.

We will see whether we can modify the system response by adding feedback. That is, we will measure the output of the circuit and apply a voltage to the system to attempt to counteract the ringing. We will denote the transfer function of our feedback system as $H(s)$.

Our closed loop now has a transfer function

$$
\begin{aligned}
T(s) &= \frac{G(s)}{1 + G(s)H(s)} \\
&= \frac{\frac{1}{s^2+0.3s+1}}{1 + \frac{1}{s^2+0.3s+1}H(s)}
\end{aligned}
$$

Feedback Example - Resonant Circuit

We will use a particular feedback system that has a transfer function $H(s) = s + 1$.

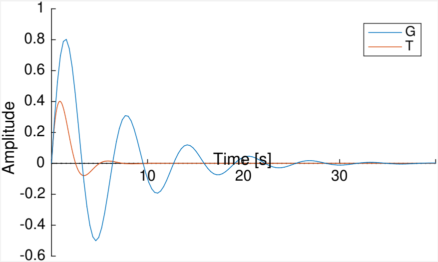
$$T(s) = \frac{\frac{1}{s^2+0.3s+1}}{1 + \frac{s+1}{s^2+0.3s+1}}$$
$$= \frac{1}{s^2 + 0.3s + s + 1 + 1}$$
$$= \frac{1}{s^2 + 1.3s + 2}$$

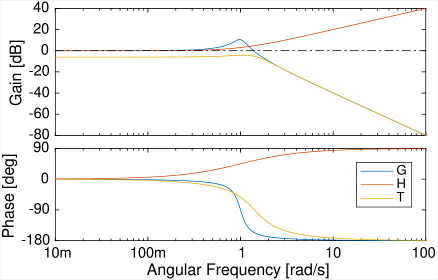
We can see that the addition of our feedback system has significantly changed the location of the system poles. In fact, the poles have moved from approximately $-0.15 \pm \mathrm{j}$ in the original system to $-0.65 \pm 1.25\mathrm{j}$.

That is, we have made the response faster and improved the damping ratio.

Feedback Example - Resonant Circuit

Let's compare our closed loop response with the original response. As expected, the oscillation dies away faster in the time domain. We also see that the resonance peak is much smaller in the frequency response.





Modes

Notice that the impulse and the step responses in the previous example both contain the same mode e^{-at}, which occurs independent of the input. We have previously seen this happen when comparing the solution of a DEs with an input and with non-zero initial conditions.

Understanding the modes is one of the most important aspects of studying any system, as it tells us how the system will respond.

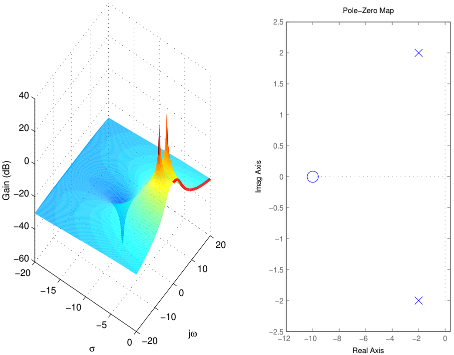
We gain insight into the modes by examining the location of the poles of the transfer function (is the values of s that make the denominator zero).

For example, the mode e^{-2t} has a corresponding representation in the s-plane of $\frac{1}{s+2}$. That is, the mode $\mathrm{e}^{-2t}\mathrm{u}(t)$ corresponds to a pole at $s = -2$.

Similarly the mode $\mathrm{e}^{-4t}\sin(3t)\mathrm{u}(t)$ has a Laplace transform of $\frac{3}{(s+4)^2+3^2}$. This corresponds to a pair of poles at $s = -4 \pm \mathrm{j}3$.

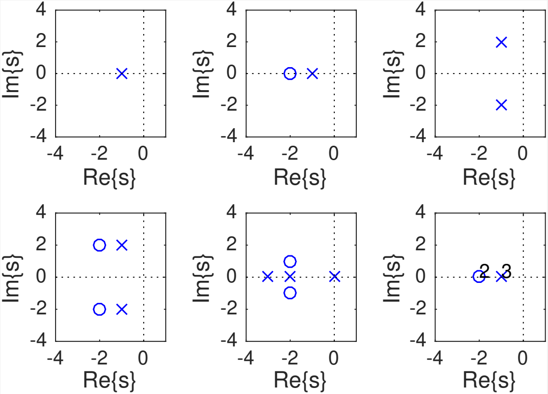
While we could plot the magnitude of a Laplace transform as a surface in the s-plane, but normally we are content with just indicating the locations of the poles and zeros (the locations that make the *numerator* transfer function zero).

Poles of a system are indicated by a cross and zeros by a circle.



Some example pole zero maps

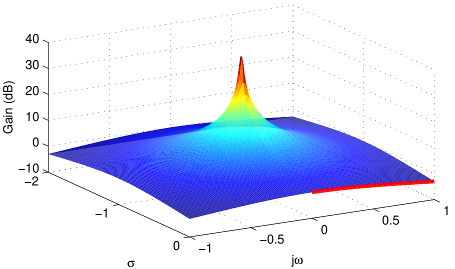
$G(s) = K\frac{1}{s+1}$ $\qquad$ $G(s) = K\frac{s+2}{s+1}$ $\qquad$ $G(s) = K\frac{1}{(s+1)^2+4}$



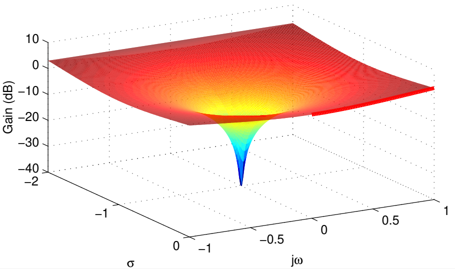
$G(s) = K\frac{(s+2)^2+4}{(s+1)^2+4}$ $\qquad$ $G(s) = K\frac{(s+2)^2+1}{s(s+2)(s+3)}$ $\qquad$ $G(s) = K\frac{(s+2)^2}{(s+1)^3}$

The s-plane

A surface plot of the s-plane should give you some sense of why poles and zeros have the names that they do...



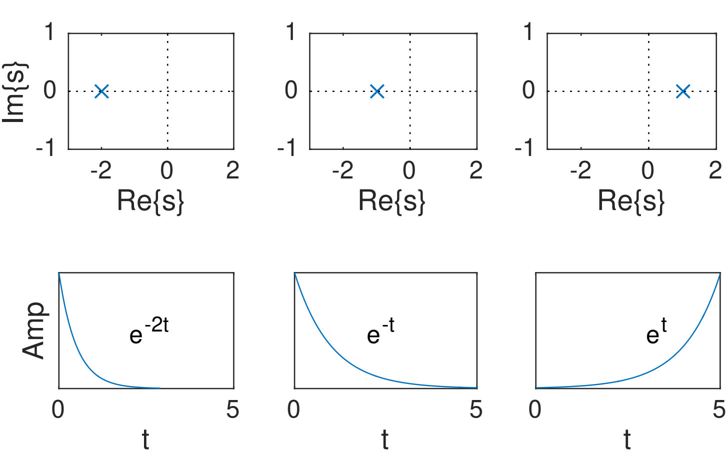
Pole



Zero

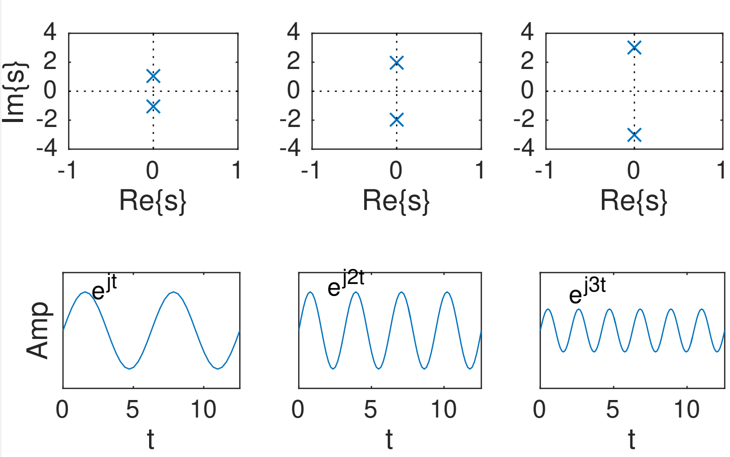
Poles on the real axis

Poles on the real axis correspond to simple exponentials. A pole at $s = a$ corresponds to a mode of e^{at}, Thus poles in the left half plane correspond to decaying exponentials and poles in the right half plane correspond to growing exponentials.



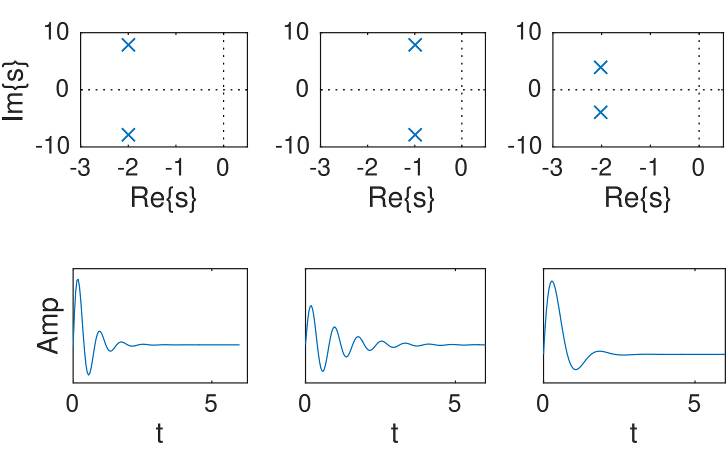
Pole pairs on the imaginary axis

A complex pair of poles on the imaginary axis correspond to a pure sinusoidal mode. Poles at $s = \pm \mathrm{j}\omega$ lead to a mode $\mathrm{e}^{\pm \mathrm{j}\omega t}$. Thus moving further away from the origin corresponds to increasing frequency.



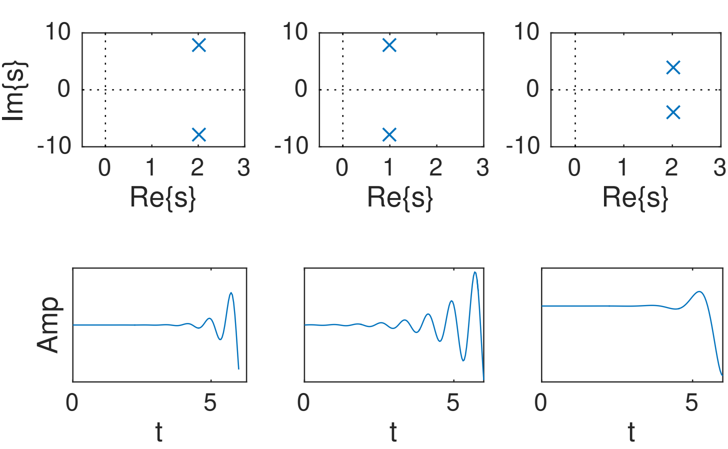
Pole pairs at arbitrary locations

Poles at other locations have an exponential envelope determined by their real part and a sinusoidal variation determined by their imaginary part.



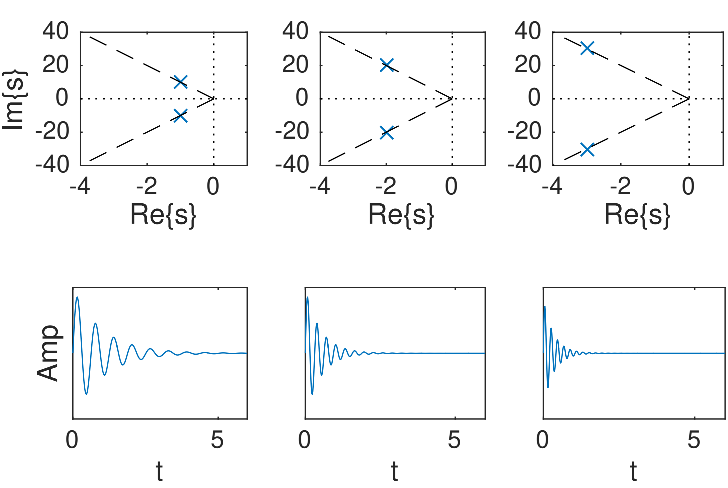
Pole pairs at complex locations

If the real part is in the right half of the s-plane then we get a growing sinusoid.

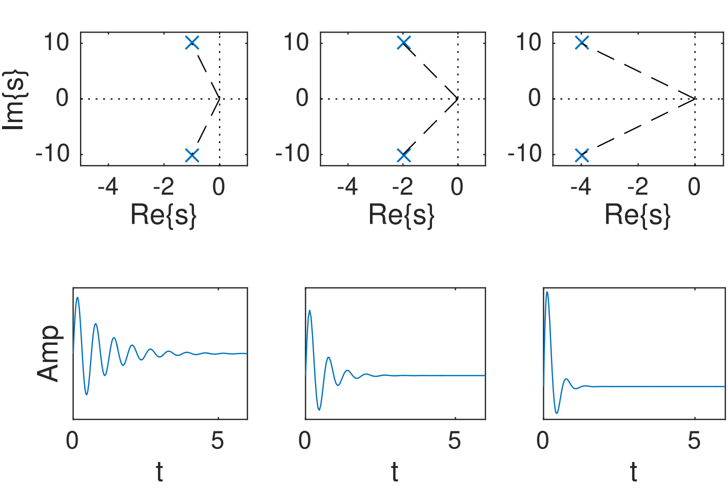


Damping ratio

Pairs of complex poles that are located along a line through the origin will have the same *damping ratio*. That is, they will have the same decay (or growth) per cycle of their sinusoidal part.



As the angle of that line changes the amount of damping changes. The damping ratio ζ is defined as the imaginary part of the pole location, divided by its real part, ie $\zeta = \frac{\omega}{\sigma}$.



Zeros

The zeros of a transfer function do not affect the modes produced by a system. However, they do play a part in determining the relative magnitude of the various modes.

If some pole is close to a zero in the s-plane, then the mode corresponding to that pole will be small.

When working on problems you will see the zeros come into problems when you are doing partial fractions expansions. The zeros are not in the denominator of a transfer function, so have no effect on its factorisation. However, they do have an affect once you are calculating the residuals.

Frequency response

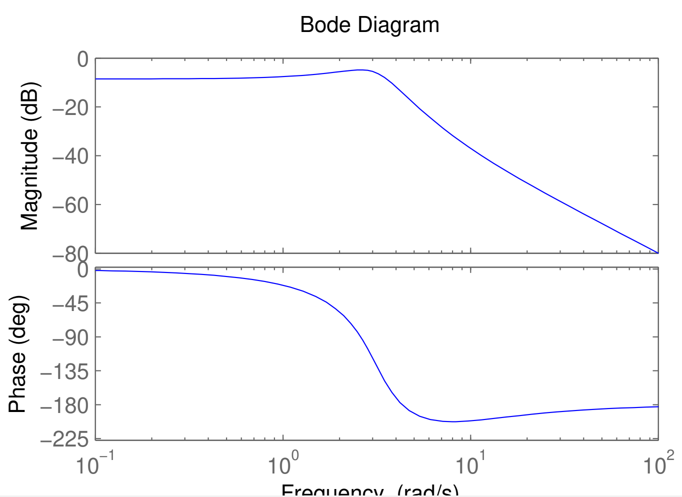
Along with the impulse and step responses, a third common experimental test for a system is the frequency response. This is conducted by driving a system with a sinusoidal signal at a variety of frequencies. We find that the system's response will (usually) depend on the frequency of the applied signal.

If we drive a linear, time-invariant system with a sinusoidal signal then we will get a sinusoidal output. In general, the output sinusoid will have a different amplitude and phase than the input. The relation between the input and output is known as the frequency response of the system.

That is, if we drive the system with a sinusoidal signal $A\mathrm{e}^{\mathrm{j}\omega t}$ and we will get an output $GA\mathrm{e}^{\mathrm{j}\omega t+\theta}$. This corresponds to the system having a gain of G and a phase shift of θ at the frequency ω.

A frequency response plot

As an example, consider the transfer function $G(s) = \frac{(s+5)(s+6)}{(s^2+4s+8)(s^2+2s+10)}$, we find the following frequency response:



Frequency response from the s-plane

The frequency response can be found by examining the s-plane along the imaginary axis ($s = \mathrm{j}\omega$, or $\sigma = 0$), as shown in the figure.