EEEN203
Circuit Analysis
Solving Differential Equations with the Laplace Transform

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Revision 93

Differentiation in the time domain

In some sense the property of the Laplace transform that is most important to us is the *differentiation in time* property. It is this property that will allow us to convert a differential equation in the time domain into an algebraic equation in the s-domain.

$$\boxed{\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0^-)}$$

where $f(0^-)$ is the value of $f(t)$ *just* before $t = 0$.

From the basic theorem we can also find higher derivatives.

$$\mathcal{L}\left\{f''(t)\right\} = s^2\mathcal{L}\left\{f(t)\right\} - sf(0^-) - f'(0^-)$$
$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0^-)\ldots - f^{(n-1)}(0^-)$$

Proof.

$$
\begin{aligned}
\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} &= \int_{0^-}^{\infty} \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \left[f(t)\mathrm{e}^{-st}\right]_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \left[f(t)\mathrm{e}^{-st}\right]_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t
\end{aligned}
$$

If the Laplace transform is to exist the first term cannot be infinite. In other words, $\lim_{t\to\infty} f(t)\mathrm{e}^{-st} = 0$.

$$
\implies \mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = -f(0^-) + sF(s)
$$

$$
\frac{\mathrm{d}f}{\mathrm{d}t} \Leftrightarrow sF(s) - f(0^-)
$$

Initial conditions at $t = 0^-$

Note here that $f(0^-)$ is the initial value of $f(t)$ and its taken *before* anything that happens at $t = 0$, so it ignores signals such as $\delta(t)$ and $\mathrm{u}(t)$.

Similarly $f^{(k)}(0^-)$ is the initial value of the k-th derivative of $f(t)$.

Solving DEs - an example

The previous theorem is the first tool we will need to solve DEs using Laplace transforms. Consider a system that has a voltage described by the DE:

$$v''(t) + 4v'(t) + 3v(t) = 0$$

with initial conditions $v(0) = 3$, $v'(0) = 1$. We want to find $v(t)$.

We start by taking the Laplace transform of the entire differential equation. Using the differentiation in time formula, we can then write the transforms of each of the derivatives of v.

$$\begin{aligned}
\mathcal{L}\left\{v'\right\} &= sV(s) - y(0) \\
&= sV - 3 \\
\mathcal{L}\left\{v''\right\} &= s^2V - sv(0) - v'(0) \\
&= s^2V - 3s - 1 \\
\therefore (s^2V - 3s - 1) + 4(sV - 3) + 3V &= 0
\end{aligned}$$

Solving DEs - an example

$$
\begin{aligned}
s^2V - 3s - 1 + 4(sV - 3) + 3V &= 0 \\
(s^2 + 4s + 3)V(s) &= 3s + 13 \\
V(s) &= \frac{3s+13}{s^2+4s+3} \\
&= \frac{3s+13}{(s+1)(s+3)} \\
&= \frac{5}{s+1} + \frac{-2}{s+3} \qquad \text{(Partial Fractions)} \\
&= 5\mathcal{L}\left\{\mathrm{e}^{-t}\mathrm{u}(t)\right\} - 2\mathcal{L}\left\{\mathrm{e}^{-3t}\mathrm{u}(t)\right\}
\end{aligned}
$$

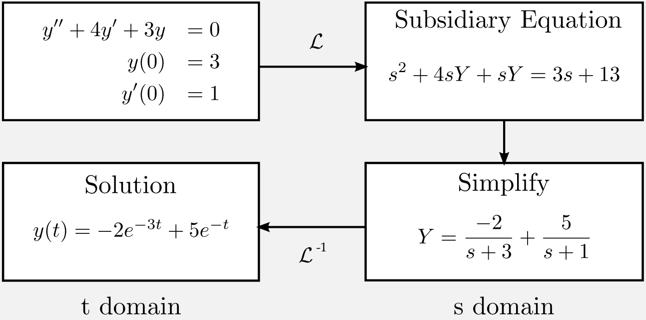
$$
\implies v(t) = \left(5\mathrm{e}^{-t} - 2\mathrm{e}^{-3t}\right)\mathrm{u}(t)
$$

We have solved our DE by Laplace transforming it, solving an algebraic equation and then transforming back to the time domain.

We shall talk more about how to use partial fractions to perform the penultimate step of the derivation.

Solving LTI DEs with the Laplace transform

The general process of using the Laplace transform to solve a DE is to move the equation from the time domain into the s-domain, which converts differential equations to algebraic equations. The algebraic equations are then solved, after which we convert back to the time domain to find the solution.



Partial fractions expansion

Before we try to inverse Laplace transform a rational function in the s domain, would like to rewrite it as the sum of expressions that we recognise. For example, if we have a rational function

$$G(s) = \frac{s^2 + 7}{s(s+2)\big((s+2)^2 + 16\big)}$$

then we would prefer to write it in a form where we recognise all of the individual terms as having simple inverse Laplace transforms.

$$G(s) = A\frac{1}{s} + B\frac{1}{s+2} + C\frac{4}{(s+2)^2 + 16}$$

This is a partial fractions expansion of the rational functions. The process consists of two parts;

1. Recognising the right constituent parts,
2. Finding the constants A, B, C (etc), which are known as the *residues* associated with the factors.

Starting point for partial fractions

The partial fractions procedure *must* begin with a strictly proper rational function. That is, we must have the function of the form

$$Y(s) = \frac{n(s)}{d(s)}$$

with $n(s)$ and $d(s)$ polynomials of s such that $n(s)$ has lower order than $d(s)$. If you have something like $Y(s) = \frac{\frac{1}{s}+1}{(s+2)(s+3)}$ you *cannot* yet use partial fractions. First put it into the required form.

$$\begin{aligned} Y(s) &= \frac{\frac{1}{s}+1}{(s+2)(s+3)} \\ &= \frac{\frac{1}{s}(s+1)}{(s+2)(s+3)} \\ &= \frac{s+1}{s(s+2)(s+3)} \end{aligned}$$

Extension - Starting point for partial fractions

Rational functions that are not strictly proper occur only rarely when analysing real systems. However, when they do occur, polynomial division can be used to rewrite them as a polynomial plus a rational remainder.

For example, consider the function $Z(s) = \dfrac{2s^2 + 9s + 6}{s^2 + 3s + 2}$. If we use rational division we obtain

$$Z(s) = 2 + \frac{3s + 2}{s^2 + 3s + 2}$$

The rational function part of this expression can now be expanded using partial fractions. We will see later how to deal with any leading polynomial terms.

Partial fractions example #1

As an example, let's find the inverse Laplace transform of $Y(s) = \dfrac{s+1}{s^3+s^2-6s}$. We start by simplifying the denominator and then separate the result into fractions, each having a denominator consisting of a first order polynomial in s.

$$\begin{aligned} Y(s) &= \frac{s+1}{s^3+s^2-6s} \\ &= \frac{s+1}{s(s^2+s-6)} \\ &= \frac{s+1}{s(s-2)(s+3)} \\ &= \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3} \end{aligned}$$

Where A,B and C are real constants that are as yet unknown.

Partial fractions example #1

The first method we will use is multiplying out the new form of the equation and equating it with the original form.

$$
\begin{aligned}
\frac{0s^2 + 1s + 1}{s^3 + s^2 - 6s} &= \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3} \\
&= \frac{A(s-2)(s+3) + Bs(s+3) + Cs(s-2)}{s^3 + s^2 - 6s} \\
&= \frac{(A+B+C)s^2 + (A+3B-2C)s - 6A}{s^3 + s^2 - 6s}
\end{aligned}
$$

The denominators of these expressions are identical, so the numerators must also be. We therefore equate coefficients of the various powers of s in the numerator polynomials of the two sides.

$$
\begin{cases}
s^2: & 0 = A + B + C \\
s^1: & 1 = A + 3B - 2C \\
s^0: & 1 = -6A
\end{cases}
$$

Partial fractions example #1

From the last of these equations, we know that $A = -\frac{1}{6}$. Substituting into the other two equations we find

$$B + C = \frac{1}{6}$$
$$3B - 2C = \frac{7}{6}$$

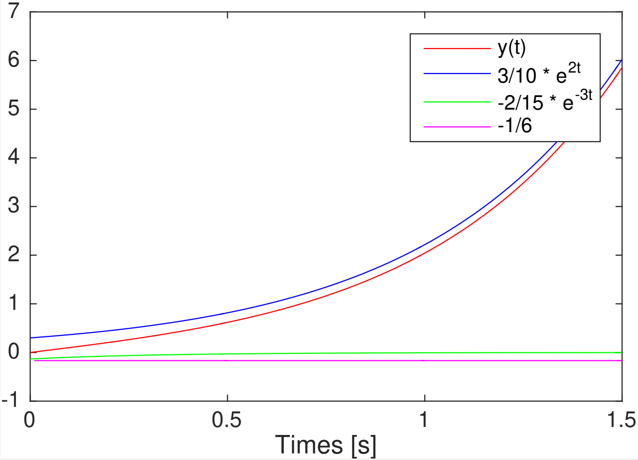
Solving these two equations simultaneously we find $B = \frac{3}{10}, C = \frac{-2}{15}$.

$$\text{So, } Y(s) = \frac{s+1}{s^3 + s^2 - 6s} = -\tfrac{1}{6}\frac{1}{s} + \tfrac{3}{10}\frac{1}{s-2} - \tfrac{2}{15}\frac{1}{s+3}$$

Taking the inverse Laplace transform we therefore find

$$y(t) = \left(-\tfrac{1}{6} + \tfrac{3}{10}\mathrm{e}^{2t} - \tfrac{2}{15}\mathrm{e}^{-3t}\right)\mathrm{u}(t)$$

Partial fractions example #1



The Heaviside or cover-up method

There is a quicker method for finding the partial fraction expansion, known as the *Heaviside* or *cover-up* method. "Cover" the term in the denominator for which you are trying to find the coefficient and then calculate the value of the remaining fraction at the value that would cause the covered term to be zero.

$$\frac{s+1}{s^3+s^2-6s} = \frac{s+1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$A = \left.\frac{s+1}{(s-2)(s+3)}\right|_{s\to 0} = -\frac{1}{6}$$

$$B = \left.\frac{s+1}{s(s+3)}\right|_{s\to 2} = \frac{3}{2\times 5} = \frac{3}{10}$$

$$C = \left.\frac{s+1}{s(s-2)}\right|_{s\to -3} = \frac{-2}{-3\times -5} = -\frac{2}{15}$$

A fun(?) mathematical aside

Technically we would be a little sloppy writing expressions like $A = \left.\dfrac{s+1}{(s-2)(s+3)}\right|_{s\to 0}$ to describe the Heaviside method.
The mathematically correct description of what we are doing is

$$A = \lim_{s\to 0} \frac{s(s+1)}{s(s-2)(s+3)}$$

Notice that in this example we have multiplied by s to cancel the pole at s. However, the catch is that

$$\frac{s(s+1)}{s(s-2)(s+3)} \neq \frac{s+1}{(s-2)(s+3)}$$

because the two expressions have different domains ($s \in \mathbb{R} \setminus 0$ vs $\mathbb{R}$). That is, the first expression is not defined at $s = 0$. But that is precisely where we would like to evaluate the function! As a result we must formally use limits in these expressions.

Unique complex factors

When confronted by a pair of complex poles $\alpha \pm j\beta$ we could expand our function to include terms like

$$\frac{C}{s-\alpha-\mathrm{j}\beta}+\frac{C^*}{s-\alpha+\mathrm{j}\beta}$$

However, it gets a bit cumbersome dealing with all the j's, and we don't need to break things down this far in any case. It is easier if we decompose our function to include a term like

$$\frac{As+B}{(s+\alpha)^2+\beta^2}$$

In general, the simplest method for solving problems like these is to use the methods above to determine the real roots and then multiply through to find A and B.

Unique complex factors example

Find the inverse Laplace transform of $Y(s) = \dfrac{1}{s(s^2+s+1)}$. First let's check whether we can factorise (s^2+s+1) in terms of real roots.

$$\Delta = 1^2 - 4.1.1 = -3 < 0$$

Therefore there are no real roots of (s^2+s+1). If there were then we would factorise it and proceed as above.

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2+s+1)} \\ &= \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2+s+1} \end{aligned}$$

Now,

$$A_1 = \left.\frac{1}{s^2+s+1}\right|_{s\to 0} = 1$$

Unique complex factors example

$$
\begin{aligned}
Y(s) = \frac{1}{s(s^2+s+1)} &= \frac{1}{s} + \frac{A_2 s + A_3}{s^2+s+1} \\
&= \frac{(s^2+s+1) + A_2 s^2 + A_3 s}{s(s^2+s+1)} \\
&= \frac{(A_2+1)s^2 + (A_3+1)s + 1}{s(s^2+s+1)}
\end{aligned}
$$

We equate the coefficients of the powers of s in the numerators of the two sides.

$$
\begin{aligned}
s^2 :&\quad 0 = A_2 + 1 \implies A_2 = -1 \\
s^1 :&\quad 0 = A_3 + 1 \implies A_3 = -1 \\
s^0 :&\quad 1 = 1
\end{aligned}
$$

Completing the square

Thus, we have $Y(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1}$.

Consider the term $\frac{s+1}{s^2+s+1}$. We would like the denominator to look like $(s+a)^2 + b$, as that is the form of our sinusoids in the table of Laplace transforms.

We know that $(s+a)^2 + b = s^2 + 2as + a^2 + b$. We can immediately find a by inspecting the coefficient of s in the denominator of our polynomial. In this case we have $a = \frac{1}{2}$.

We can equate the constants too, to find $a^2 + b = 1 \implies b = \frac{3}{4}$.

Splitting into (co)sinusoidal parts

So, we have

$$\frac{s+1}{s^2+s+1} = \frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

We now want to split the function into a part that "looks" like a decaying cosinusoid and a part that "looks" like a decaying sinusoid.

Recall that a sinusoid has a numerator that matches the squared term in the denominator ($s+\frac{1}{2}$ in this case). We can just split the polynomial into two parts;

$$\frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{s+\frac{1}{2}+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Rescaling the sinusoidal part

We would like decaying sinusoidal part to look like its prototype $\frac{\omega}{(s+a)^2+\omega^2}$. We need to make the numerator into ω.

We know from the denominator that $\omega^2 = \frac{3}{4}$, so in this case $\omega = \frac{\sqrt{3}}{2}$.

Therefore, let's write the decaying sinusoidal term as

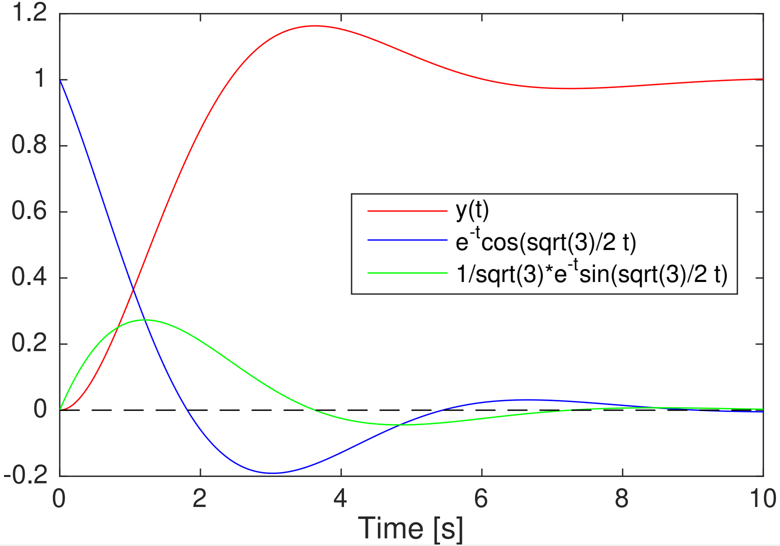
$$\frac{\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Reassembling everything gives us

$$Y(s) = \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$\implies y(t) = \left[1 - \mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\mathrm{u}(t)$$

Unique complex factors example



A second order ODE revisited

Let's reconsider a second order differential equation and see how the behaviour of the output changes as we alter one of the coefficients. We met this equation before, although it had different initial conditions on that occasion.

$$\begin{aligned} \ddot{y} + 4\dot{y} + 3y &= 0, \quad \text{with } y(0) = 1, \dot{y}(0) = 0 \\ \implies s^2Y + 4sY + 3Y &= sy(0) + 4y(0) \\ Y(s) &= \frac{s+4}{s^2+4s+3} \end{aligned}$$

We can factor the denominator using real roots. If you used the quadratic equation you would find $\Delta > 0$.

$$\begin{aligned} Y(s) &= \frac{3}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3} \\ \therefore y(t) &= \left[\frac{3}{2}\mathrm{e}^{-t} - \frac{1}{2}\mathrm{e}^{-3t}\right]\mathrm{u}(t) \end{aligned}$$

A second order ODE revisited

Let's alter the equation slightly and solve it again. We will change the coefficient of y.

$$\begin{aligned} \ddot{y} + 4\dot{y} + 5y &= 0, \quad \text{with } y(0) = 1, \dot{y}(0) = 0 \\ \implies s^2Y + 4sY + 5Y &= sy(0) + 4y(0) \\ Y &= \frac{s+4}{s^2+4s+5} \end{aligned}$$

The denominator cannot be factored into real roots this time, ie $\Delta < 0$.

$$\begin{aligned} Y &= \frac{s+4}{(s+2)^2+1} \\ &= \frac{s+2}{(s+2)^2+1} + \frac{2}{(s+2)^2+1} \\ \therefore y(t) &= \left[\mathrm{e}^{-2t}\cos(t) + 2\mathrm{e}^{-2t}\sin(t)\right]\mathrm{u}(t) \end{aligned}$$

A second order ODE revisited

We haven't yet considered the case where $\Delta = 0$. As an example, this occurs for the differential equation

$$\begin{aligned} \ddot{y} + 4\dot{y} + 4y &= 0, \quad \text{with } y(0) = 1, \dot{y}(0) = 0 \\ \implies s^2Y + 4sY + 4Y &= sy(0) + 4y(0) \\ Y &= \frac{s+4}{s^2+4s+4} \end{aligned}$$

The denominator can be factored into a *repeated* real root, ie $\Delta = 0$.

$$\begin{aligned} Y &= \frac{s+4}{(s+2)^2} \\ &= \frac{2}{(s+2)^2} + \frac{1}{s+2} \\ \therefore y(t) &= \left[\mathrm{e}^{-2t} + 2\mathcal{L}^{-1}\left\{ \frac{1}{(s+2)^2} \right\} \right] \mathrm{u}(t) \end{aligned}$$

We have a problem, as we don't know how to find $\mathcal{L}^{-1}\left\{\frac{c}{(s+a)^2}\right\}$.

A second order ODE revisited

This is still a second order differential equation, and as such is required to have two independent solutions. You may recall that differential equations like this can be solved with a second solution of form te^{-at}.

That is, for our particular DE we expect a solution that looks like

$$y(t) = \mathrm{e}^{-2t} + ct\mathrm{e}^{-2t} \quad \text{for some } c \in \mathbb{R}$$

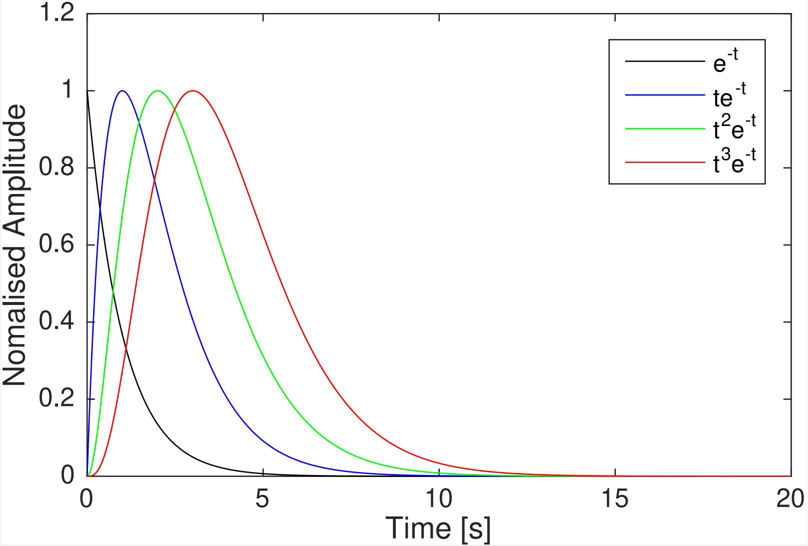
We should therefore expect that there will be a link between t-domain expression like $t\mathrm{e}^{-at}$ and s-domain expressions that look like $\frac{1}{(s+a)^2}$.

Modes associated with repeated roots

We have previously claimed that (almost) the only solutions to constant coefficient, linear differential equations were complex exponentials. The existence of these extra modes is the reason for the “almost”. In fact, in the case of repeated roots, modes made of a polynomial of t mulitplied by a complex exponential are possible. eg. $t\mathrm{e}^{-at}$, $t^2\cos(\omega t)$.

It is worth thinking about the form of the $t^n\mathrm{e}^{-at}$ modes a little. For large t the exponential decay will dominate. But for small t they will look like their polynomial part. Notice that the modes become more sluggish as n increases.

Modes associated with repeated roots



Laplace transform of $tf(t)$

We are going to take a slightly roundabout way to finding the Laplace transform of functions that look like $tf(t)$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}F(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_{0^-}^{\infty} \frac{\partial}{\partial s} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_{0^-}^{\infty} f(t)\frac{\partial}{\partial s}\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_{0^-}^{\infty} -tf(t)\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \mathcal{L}\left\{-tf(t)\right\}
\end{aligned}$$

We have just proved the *differentiation in the s domain* property of the Laplace transform.

$$\boxed{tf(t) \Longleftrightarrow -F'(s)}$$

$\mathcal{L}\left\{t^n e^{-at}\right\}$

We can use this property find the Laplace transform of functions with form $t^n \mathrm{e}^{-at}$.

$$\begin{aligned}
\mathcal{L}\left\{t\mathrm{e}^{-at}\right\} &= -\frac{\mathrm{d}\mathcal{L}\left\{\mathrm{e}^{-at}\right\}}{\mathrm{d}s} \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s+a} \\
&= \frac{1}{(s+a)^2} \\
t\mathrm{e}^{-at} &\Longleftrightarrow \frac{1}{(s+a)^2}
\end{aligned}$$

We can repeat this argument as many times as we like, to find the general formula for roots repeated more than twice.

$$t^{n-1}\mathrm{e}^{-at} \Longleftrightarrow \frac{n!}{(s+a)^n}$$

A second order ODE revisited (again!)

We can now return to the solution of
$\ddot{y} + 4\dot{y} + 4y = 0, \quad$ with $y(0) = 1, \dot{y}(0) = 0$ which we abandoned after reaching

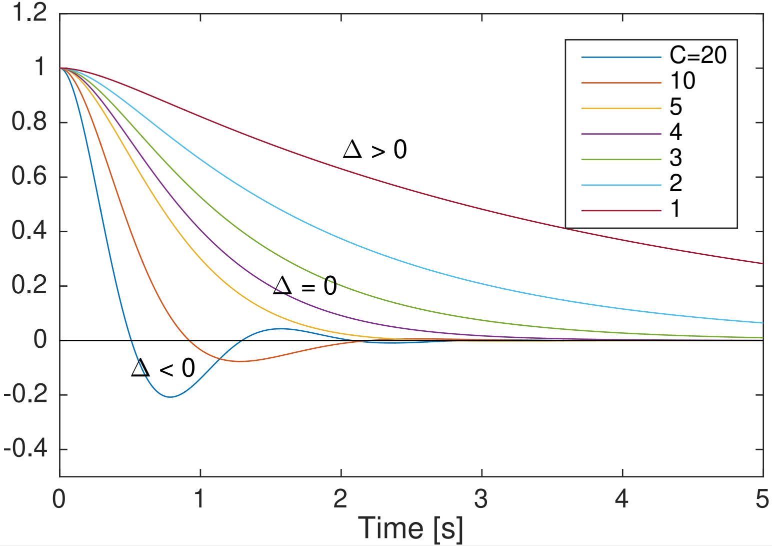
$$y(t) = \left[\mathrm{e}^{-2t} + 2\mathcal{L}^{-1}\left\{ \frac{1}{(s+2)^2} \right\} \right] \mathrm{u}(t)$$

We can now conclude this inverse Laplace transform.

$$y(t) = \left[\mathrm{e}^{-2t} + 2t\mathrm{e}^{-2t} \right] \mathrm{u}(t)$$

It may seem a bit startling that we get dramatically different functional forms by changing a single coefficient smoothly. It may seem particularly strange to have a completely different form for the precise case of the repeated root, when a denominator of $(s + 1.999)(s + 2.001)$ would be completely different. However, in practice we actually have a smooth change in the graph as we move the repeated root.

A second order ODE revisited (again!)



Repeated factors

Now that we know the meaning of repeated factors in the s-domain, we can determine how to inverse Laplace transform them back into the time domain. Consider a rational function $Y(s)$ containing repeated roots at $s = a$ in its denominator.

$$Y(s) = \frac{\dots}{\dots \times (s-a)^n}$$

When we decompose this system with partial fractions we get a series of terms corresponding to all powers of $(s-a)$ less than or equal to n.

$$Y(s) = \dots + \frac{c_n}{(s-a)^n} + \frac{c_{n-1}}{(s-a)^{n-1}} + \dots + \frac{c_1}{s-a}$$

(for $c_i \in \mathbb{R}$)

Example:

$$\frac{s+7}{(s+2)^3(s+6)} = \frac{A_3}{(s+2)^3} + \frac{A_2}{(s+2)^2} + \frac{A_1}{s+2} + \frac{B}{s+6}$$

Repeated factors

If the denominator of our fraction contains repeated roots then the Heaviside method breaks down when searching for the coefficients of the terms that have multiple roots.

Consider a partial fraction expansion of $Y(s)$ that includes terms of form $\dfrac{A_m}{(s-a)^m} + \dfrac{A_{m-1}}{(s-a)^{m-1}} + \ldots + \dfrac{A_1}{s-a}$ for some a.

It can be shown that, for $k \neq m$

$$A_m = \lim_{s \to a} (s-a)^m Y(s) \qquad \text{(just the cover-up method again)}$$

$$A_k = \frac{1}{(m-k)!} \lim_{s \to a} \frac{\mathrm{d}^{m-k}}{\mathrm{d}s^{m-k}} (s-a)^m Y(s)$$

Repeated factors

You should recognise the equation for A_m as being a mathematical description of the cover up method. That is, we can use the normal cover up method for finding the coefficient associated with the highest power of the repeated root.

The other terms look ominous, but are straightforward if you are comfortable differentiating rational functions.

Repeated factors example

$$Y(s) = \frac{3s+8}{(s+2)^2} = \frac{A_2}{(s+2)^2} + \frac{A_1}{s+2}$$

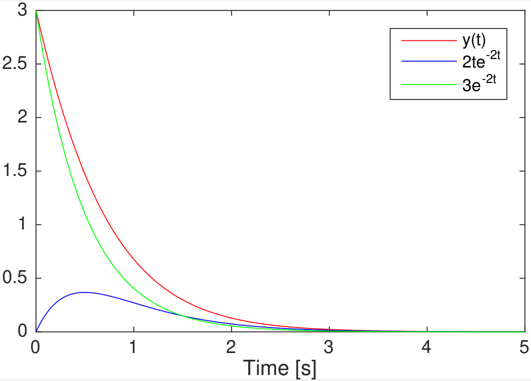
$$\begin{aligned} A_2 &= \lim_{s\to -2} \frac{(s+2)^2 \cdot (3s+8)}{(s+2)^2} \\ &= 3s+8\big|_{s\to -2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} A_1 &= \frac{1}{(2-1)!} \lim_{s\to -2} \frac{\mathrm{d}}{\mathrm{d}s} \frac{(s+2)^2 \cdot (3s+8)}{(s+2)^2} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}(3s+8)\bigg|_{s\to -2} \\ &= 3 \end{aligned}$$

Repeated factors example #1

Therefore, we have found

$$Y(s) = \frac{2}{(s+2)^2} + \frac{3}{s+2}$$
$$\implies y(t) = \left[2t\mathrm{e}^{-2t} + 3\mathrm{e}^{-2t}\right] \mathrm{u}(t)$$



Repeated factors example #2

Find the inverse Laplace transform of $G(s)$ where

$$\begin{aligned} G(s) &= \frac{7s^2 + 45s + 66}{(s+2)^2(s+6)} \\ &= \frac{A_2}{(s+2)^2} + \frac{A_1}{s+2} + \frac{B}{s+6} \end{aligned}$$

Let's use the Heaviside technique to find A_2 and B.

$$\begin{aligned} B &= \left. \frac{7s^2 + 45s + 66}{(s+2)^2} \right|_{s=-6} = \frac{48}{16} = 3 \\ A_2 &= \left. \frac{7s^2 + 45s + 66}{s+6} \right|_{s=-2} = \frac{4}{4} = 1 \end{aligned}$$

Repeated factors example #2a

Let's find A_1 using the formula

$$\begin{aligned}
A_1 &= \frac{1}{(2-1)!}\lim_{s\to -2}\frac{\mathrm{d}}{\mathrm{d}s}\frac{7s^2+45s+66}{s+6} \\
&= \lim_{s\to -2}\frac{(14s+45)(s+6)-(7s^2+45s+66)(1)}{(s+6)^2} \quad \text{(Quotient rule)} \\
&= \frac{4(-28+45)-(28-90+66)}{4^2} \\
&= \frac{64}{16} \\
&= 4
\end{aligned}$$

So, $G(s) = \dfrac{1}{(s+2)^2} + \dfrac{4}{s+2} + \dfrac{3}{s+6}$

$$\rightsquigarrow g(t) = \left[t\mathrm{e}^{-2t} + 4\mathrm{e}^{-2t} + 3\mathrm{e}^{-6t}\right]\mathrm{u}(t)$$

Repeated factors example #2b

As an alternative, we could avoid differentiating the rational function by multiplying out once we have found A_2 and B.

$$\begin{aligned}
G(s) &= \frac{7s^2 + 45s + 66}{(s+2)^2(s+6)} \\
&= \frac{1}{(s+2)^2} + \frac{A_1}{s+2} + \frac{3}{s+6} \\
&= \frac{(s+6) + A_1(s+2)(s+6) + 3(s+2)^2}{(s+2)^2(s+6)} \\
&= \frac{(A_1+3)s^2 + (13+8A_1)s + 18 + 12A_1}{(s+2)^2(s+6)}
\end{aligned}$$

Equating any coefficient of s yields $A_1 = 4$ as before. Note that you don't actually have to do the complete gathering up of numerator terms in the last line. You only need one equation to establish A_1, so you could just write down the coefficient of s^2, or the constants and equate those if you chose.

One unknown coefficient

There is a useful time saving partial fractions trick that we can use once we know *all but one* coefficient. We can then use a short cut. Consider the example

$$Y(s) = \frac{2}{(s+2)^2} + \frac{A_1}{s+2}$$

This expression must be true for all values of s, except where the function is undefined. That is, the expression for $Y(s)$ must be true for any s we like, except $s = -2$. If we substitute $s = 0$ (say) we will be able to find A_1 immediately.

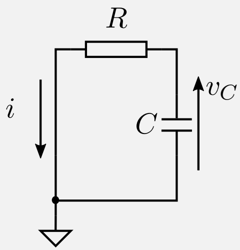
$$\frac{3s+8}{(s+2)^2} = \frac{2}{(s+2)^2} + \frac{A_1}{s+2} \implies \frac{3 \times 0 + 8}{(0+2)^2} = \frac{2}{(0+2)^2} + \frac{A_1}{0+2}$$

$$\frac{8}{4} = \frac{2}{4} + \frac{A_1}{2}$$

So, $A_1 = 3$.

RC circuit example

Let's return to the the RC circuit that we previously examined as shown in the figure having some initial capacitor voltage $v_C(0) = 2$ V.



We previously found that $\dfrac{v_C}{R} = -C\dfrac{\mathrm{d}v_C}{\mathrm{d}t}$

$$\implies \frac{\mathrm{d}v_C}{\mathrm{d}t} + \frac{1}{RC}v_C = 0$$

Let's take the Laplace transform of this equation

$$sV_C(s) - v_C(0^-) + \frac{1}{RC}V_C = 0$$

$$\implies \left(s + \frac{1}{RC}\right)V_C(s) = v_C(0^-)$$

RC circuit example

We can substitute for the initial condition ($v_C = 2$) and rearrange to get

$$V_C(s) = \frac{2}{s + \frac{1}{RC}}$$

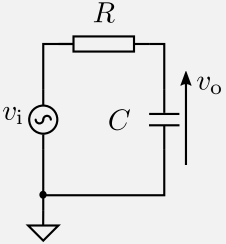
We have now solved for the capacitor voltage in the s-domain. We now only need to convert back into the time domain to solve our initial problem. Remember that $\mathrm{e}^{-at} \Leftrightarrow \frac{1}{s+a}$, so

$$v_C(t) = 2\mathrm{e}^{-\frac{t}{RC}}\mathrm{u}(t)$$

This agrees with our previous solution of the problem.

Solving a DE that includes a source

If we have a source then we simply need to find the Laplace transform of that as well when we move into the s-domain.



Consider our RC low pass filter again as shown in the figure, this time with a source included. We will assume that $v_{\mathrm{i}}(t) = \mathrm{e}^{3t}\mathrm{u}(t)$ and that there is no initial charge on the capacitor.

Kirchoff's current rule gives

$$-C\frac{\mathrm{d}v_{\mathrm{o}}}{\mathrm{d}t} + \frac{v_{\mathrm{i}} - v_{\mathrm{o}}}{R} = 0$$

$$\implies \frac{\mathrm{d}v_{\mathrm{o}}}{\mathrm{d}t} + \frac{1}{RC}v_{\mathrm{o}} = \frac{1}{RC}v_{\mathrm{i}}$$

Solving a DE that includes a source

We now Laplace transform both sides of this equation.

$$
\begin{aligned}
\mathcal{L}\left\{\frac{\mathrm{d}v_{\mathrm{o}}}{\mathrm{d}t}\right\} + \mathcal{L}\left\{\frac{1}{RC}v_{\mathrm{o}}\right\} &= \frac{1}{RC}\mathcal{L}\left\{v_{\mathrm{i}}\right\} \\
sV_{\mathrm{o}} - v_{\mathrm{o}}(0^-) + \frac{1}{RC}V_{\mathrm{o}} &= \frac{1}{RC}\mathcal{L}\left\{\mathrm{e}^{3t}\right\} \\
\left(s + \frac{1}{RC}\right)V_{\mathrm{o}} &= \frac{1}{RC}\frac{1}{s-3} \\
V_{\mathrm{o}} &= \frac{1}{RC}\frac{1}{(s-3)\left(s+\frac{1}{RC}\right)}
\end{aligned}
$$

Solving a DE that includes a source

We have now found the solution in the s-domain, but we don't know how to inverse transform this expression to get back to the time domain. We need to rearrange this expression into a form where we know the inverse Laplace transform of the individual parts.

Let's write our solution as $\dfrac{A}{s-3} + \dfrac{B}{s+\frac{1}{RC}}$ for some unknown $A, B \in \mathbb{R}$

$$
\begin{aligned}
\implies \frac{\frac{1}{RC}}{(s-3)\left(s+\frac{1}{RC}\right)} &= \frac{A}{s-3} + \frac{B}{s+\frac{1}{RC}} \\
&= \frac{As + \frac{A}{RC} + Bs - 3B}{(s-3)\left(s+\frac{1}{RC}\right)} \\
&= \frac{(A+B)s + \frac{A}{RC} - 3B}{(s-3)\left(s+\frac{1}{RC}\right)}
\end{aligned}
$$

Solving a DE that includes a source

The coefficients of the powers of s in the numerators of these expressions must be equal.

$$\implies \begin{cases} s^1: & A + B &= 0 \\ s^0: & \frac{A}{RC} - 3B &= \frac{1}{RC} \end{cases}$$

$$\text{So, } B = -A \text{ which implies } \frac{A}{RC} + 3A = \frac{1}{RC}$$

$$\left(3 + \frac{1}{RC}\right) A = \frac{1}{RC}$$

$$\therefore A = \frac{\frac{1}{RC}}{3 + \frac{1}{RC}}$$

$$A = \frac{1}{3RC + 1} \text{ and } B = -\frac{1}{3RC + 1}$$

Solving a DE that includes a source

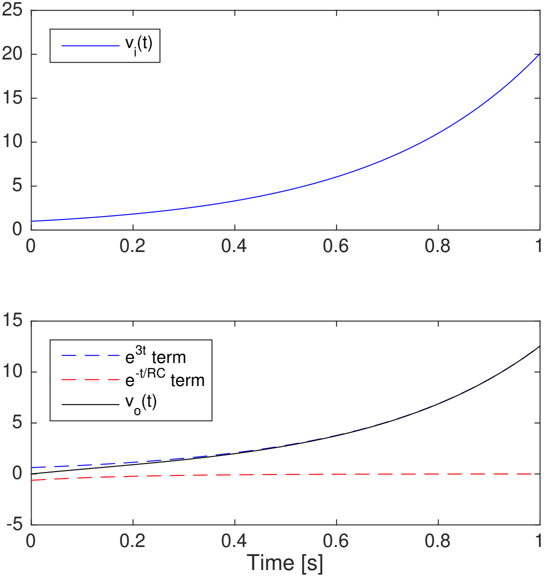
Substituting these values back into our expression for V_o, we have

$$V_\mathrm{o}(s) = \frac{1}{3RC+1} \times \frac{1}{s-3} \quad - \quad \frac{1}{3RC+1} \times \frac{1}{s+\frac{1}{RC}}$$

The expression is now made of terms for which we know the inverse Laplace transform. We can therefore find the final solution.

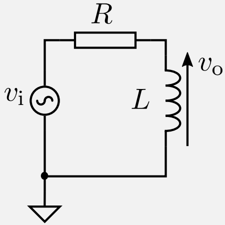
$$v_\mathrm{o}(t) = \frac{1}{3RC+1}\mathrm{e}^{3t} - \frac{1}{3RC+1}\mathrm{e}^{-\frac{t}{RC}}$$

Solving a DE that includes a source – $R = 1$, $C = 0.2$



An RL circuit with a source and initial conditions.

We can also use our method when we have both sources and initial conditions to consider.



The RL circuit shown in the figure is driven by a source $v_\mathrm{i} = \mathrm{e}^{5t}\mathrm{u}(t)$ and has $v_\mathrm{o}(0) = 2$ V. Let's solve for $v_\mathrm{o}(t)$. In this case we will find a DE for $i(t)$, so we will first solve for $i(t)$ and use that result to calculate $v_\mathrm{o} = v_\mathrm{i} - Ri(t)$.

$$v_\mathrm{i} = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t}$$
$$\implies V_\mathrm{i} = RI(s) + sLI(s) - Li(0^-)$$

An RL circuit with a source and initial conditions.

How can we find $i(0^-)$ in this case? Let's begin by defining positive i as flowing clockwise through the circuit (so that the v_o is positive in the direction shown in the diagram).

At $t = 0$, we have $v_\mathrm{i} = \mathrm{e}^0 = 1$ V and $v_\mathrm{o} = 2$ V (the initial condition). That is, we have $i(0) = \frac{1-2}{R} = -\frac{1}{R}$ amperes.

$$\text{Therefore, } (R + sL)I(s) = V_\mathrm{i} - \frac{L}{R}$$

$$I(s) = \underbrace{\frac{V_\mathrm{i}}{R + sL}}_{\text{Source}} - \underbrace{\frac{L}{R}\frac{1}{R + sL}}_{\text{Init.Cond.}}$$

$$= \frac{1}{s-5}\frac{1}{R + sL} - \frac{L}{R}\frac{1}{R + sL}$$

An RL circuit with a source and initial conditions.

Again, we need to rearrange the multiplication in the first term into a summation. Following the procedure that we used in the previous example, we end up with

$$I(s) = \tfrac{1}{5L+R}\frac{1}{s-5} - \tfrac{L}{R+5L}\frac{1}{R+sL} - \tfrac{L}{R}\frac{1}{R+sL}$$

Let's make the $\frac{1}{R+sL}$ terms look like $\frac{1}{s+a}$, which we know how to inverse transform.

$$\begin{aligned} &= \tfrac{1}{R+5L}\frac{1}{s-5} - \tfrac{L}{R+5L}\frac{\frac{1}{L}}{s+\frac{R}{L}} - \tfrac{1}{R}\frac{1}{s+\frac{R}{L}} \\ \implies i(t) &= \frac{1}{R+5L}\mathrm{e}^{5t} - \frac{L}{R+5L}\mathrm{e}^{-\frac{R}{L}t} - \frac{1}{R}\mathrm{e}^{-\frac{R}{L}t} \\ &= \frac{1}{R+5L}\mathrm{e}^{5t} - \left(\frac{R+RL+5L}{R(R+5L)}\right)\mathrm{e}^{-\frac{R}{L}t} \end{aligned}$$

An RL circuit with a source and initial conditions.

Finally, recall that

$$
\begin{aligned}
v_{\mathrm{o}}(t) &= v_{\mathrm{i}}(t) - Ri(t) \\
&= \mathrm{e}^{5t} - \frac{R}{R+5L}\mathrm{e}^{5t} + \left(\frac{R+RL+5L}{R+5L}\right)\mathrm{e}^{-\frac{R}{L}t} \\
v_{\mathrm{o}}(t) &= \left(1 - \frac{R}{R+5L}\right)\mathrm{e}^{5t} + \left(\frac{R+RL+5L}{R+5L}\right)\mathrm{e}^{-\frac{R}{L}t} \\
v_{\mathrm{o}}(t) &= \left(\frac{5L}{R+5L}\right)\mathrm{e}^{5t} + \left(\frac{R+RL+5L}{R+5L}\right)\mathrm{e}^{-\frac{R}{L}t}
\end{aligned}
$$