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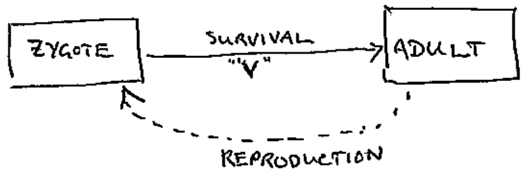
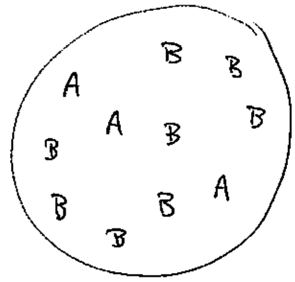
Mathematical Models of Social Evolution.

Ch 1. Viability Selection

to model we need:
- population ⟵ could model structure but we won't yet: "random mixing"
- heritable variants ⟷ 2 types, A & B
- life cycle ⟷ We'll assume strict 'generations', and:



n individuals



p = fraction of A's
"frequency".

V(A) = prob. an A survive to adulthood.

V(B) = same for B...

Say all adults produce same # offspring (for now).

Q: What is p', the freq. of A's in next generation? (zygotes)

$$p' = \frac{\#\text{ A adults}}{\text{total }\#\text{ adults}} = \frac{\cancel{n} p\, V(A)}{\cancel{n} p\, V(A) + \cancel{n}(1-p)\, V(B)}$$

$$p' = \frac{p\, V(A)}{p\, V(A) + (1-p)\, V(B)}$$

A RECURSION: Given V(A) and V(B), this equation gets you from one p to the next p, one generation lat

Changes: $\Delta p = p' - p = \ldots(\text{algebra})\ldots$

$$\Delta p = \frac{p(1-p)\,(V(A) - V(B))}{\bar{w}}$$

average fitness.

$\bar{w} = p\,V(A) + (1-p)\,V(B)$

(Same denominator as p' eq

Interested in equilibria: p for which $\Delta p = 0$.

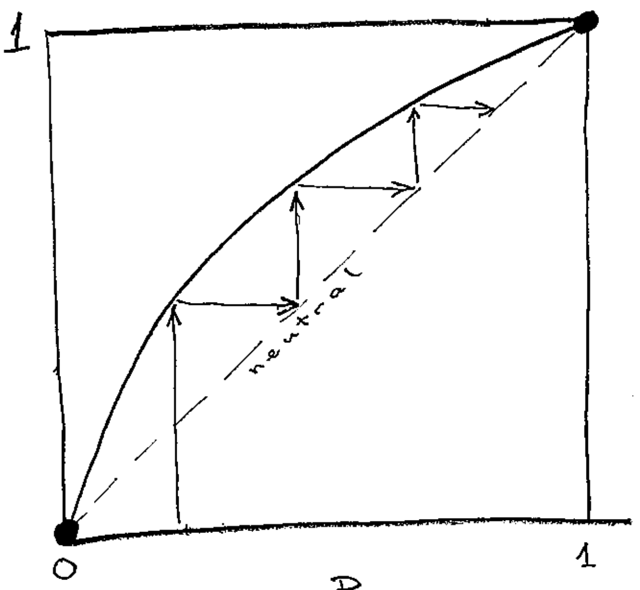
"fixed points" of the dynamics.

There are 3:
- $p = 0$
- $p = 1$
- $V(A) = V(B)$

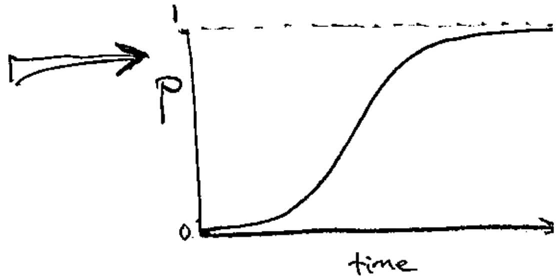
Stable? Unstable? Consider Perturbations around the equilibria: do they 'run away'

Plot p' against p (NB. I'm assuming $V(A) > V(B)$ here)

p'



p



<u>equilibrium:</u> ($p' = p$)

$p = 0$ UNSTABLE (goes up) ← Graphically the reason is that the slope is > 1, since that means it must go above the diagonal.

$p = 1$ STABLE ← Graphically: slope < 1

"A" is an ESS
Evolutionarily Stable Strategy

$V(A) = V(B)$ NEUTRAL ← $p' = p$ everywhere. Any p ~~value~~ frequency is equilibrium.

Technically:

$$\text{slope} = \frac{dp'}{dp} = \frac{V(A)\,V(B)}{\bar{w}^2}$$

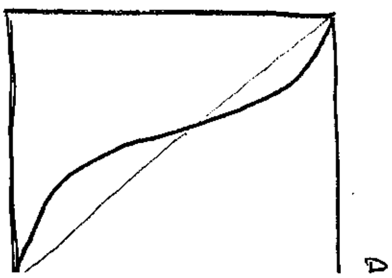
after a bit of calculus.

KEY STEPS:

- ★ Find the equilibria.
- ★ Evaluate slope $\frac{dp'}{dp}$ at each equilibrium to determine its stability.

Ex: what about ...

p'



?

Ch2: Animal Conflict — Hawks & Doves

- pairwise conflicts over some resource ("v")
- can fight, display, or run away.
- consider genetically pre-determined strategies.
- assume "fair" contests are decided (won) at random (flip a coin...)

HAWK	DOVE
Always fights. Winner gets v, Loser gets $-c$ (suffers)	Displays, but runs away if attacked.

PAYOFF MATRIX:

$$V(\text{row player} \mid \text{column player})$$

"given"

	HAWK	DOVE
HAWK	$\frac{v-c}{2}$	v
DOVE	0	$\frac{v}{2}$

e.g. $V(H|D) = v$

$V(D|H) = 0$

etc...

EXPECTED FITNESS (assuming RANDOM pairings):

Average fitness of Hawks:

$$W(H) = \underbrace{w_0}_{\text{base fitness}} + \underbrace{p}_{\text{freq. of hawks}} V(H|H) + \underbrace{(1-p)}_{\text{freq. of doves}} V(H|D)$$

$$= w_0 + p\frac{(v-c)}{2} + (1-p)v$$

Average fitness of Doves:

$$W(D) = w_0 + (1-p)\frac{v}{2}$$

AND JUST AS BEFORE:

$$p' = \frac{p\,W(H)}{p\,W(H) + (1-p)\,W(D)}$$

DYNAMICS:

$$\Delta p = p(1-p)\frac{W(H) - W(D)}{\bar{w}}$$

∴ there must be (≥ 3 equilibria to consider.

NB: Now W depend

★ is any strategy stable against invasion by others? ("Pure ESS")

...★ if not, is there a stable mixture? ("Mixed ESS")

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eqlb^n I: $p \approx 0$. Doves everywhere. Can Hawks invade?

$$\left.\begin{array}{l} W(H) \approx W_0 + V \\ W(D) \approx W_0 + V/2 \end{array}\right\} \quad W(H) > W(D)$$

~~always~~ regardless of V value

$\Longrightarrow$ Hawks invade.

eqlb^n II: $p \approx 1$. Hawks everywhere. Can Doves invade?

$$W(H) \approx W_0 + \frac{V-C}{2}$$

$$W(D) \approx W_0$$

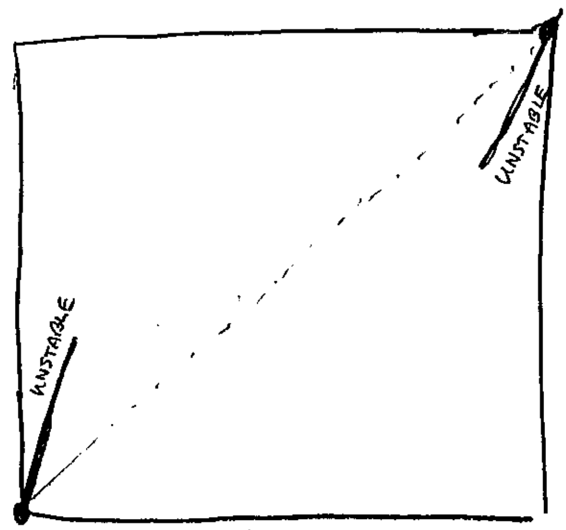
never wins resource, but never beaten up either

$\Longrightarrow$ Doves can invade iff $W(D) > W(H)$,

ie. if $\boxed{V < C}$

NOTE: together these mean that, if $V < C$, rare H can invade ~~often~~ pure D AND VICE VERSA ....

∴ there must be some stable eqlb^n mixture:



p'

p

eqlb^n III: $W(H) = W(D)$

Solve for p

$$\boxed{\begin{array}{c} p = V/C \\ \text{This MUST be an ESS.} \\ \text{Called a "mixed equilibrium"} \end{array}}$$

★ $\boxed{\text{RETALIATOR}}$ a 3rd strategy:

Plays Hawk against a Hawk
" Dove " " Dove
" " " self...

$$\left.\begin{aligned} p &= \text{freq. of Hawks.} \\ q &= \text{freq. of Retaliators.} \\ 1-p-q &= \text{freq. of Doves.} \end{aligned}\right\}$$

Ex: fill in the payoff matrix;

	H	D	R
H	?	?	?
D	?	?	?
R	?	?	?

leads to.

$$W(H) = W_0 + (p+q)\frac{V-C}{2} + (1-p-q)V$$

$$W(D) = W_0 + (1-p)\frac{V}{2}$$

$$W(R) = W_0 + p\left(\frac{V-C}{2}\right) + (1-p)\frac{V}{2}$$

START: LOOK AT THE PURE STRATEGIES.
ARE THEY INVADABLE?

all Doves $\longrightarrow$ we already know H can invade.

all Hawks $\longrightarrow$ " " " D " " (if $V < C$)

all Retaliators $\longrightarrow$ resist Hawks when $W(R|R) > W(H|R)$

$\dashrightarrow$ $C > 0$

ie Always $\Rightarrow$ "Retaliators safe from Hawks...." ?

resist Doves ? No: $W(R|R) = W(D|R)$

$\therefore$ Neutral drift

Any mix of (just) Doves and Retaliators is an equilibrium.

$\Longrightarrow$ H can invade if density of R gets too low.

how low ?

If hawks are rare, we have

$$W(R) = W(D) = W_0 + v/2.$$

f. A rare hawk has fitness

$$W(H) = w_0 + q\frac{(v-c)}{2} + (1-q)v$$

Solve for the q at which we get $W(H) > W(\underset{\text{or } D}{R})$

$\Rightarrow$ Hawks invade if $q < \dfrac{v}{v+c}$

What happens next ?

Check whether R can invade the (old) H/D mixed equilibrium.

If $v < c$ it can't !

So retaliators don't change anything ...

★ $\boxed{\text{OWNERSHIP}}$ An essentially arbitrary asymmetry ...

Consider "Bourgeois" — plays H if it arrives first
" D " " 2nd.

& this is just random.

~~Is Pure Bourgeois population an ESS??~~

$$V(B|B) = \frac{v}{2}$$ (no fights against self!).

$$V(H|B) = \frac{1}{2}\left(\frac{v-c}{2}\right) + \frac{1}{2}v$$

$$V(D|B) = \frac{1}{2}\left(\frac{v}{2}\right)$$

New entries in the Payoff Matrix :

	H	D	B
H	✓	✓	*
D	✓	✓	*
B	*	*	*

Easy to show Bourgeois can invade pure Hawk eqlb^m. ✓
or pure Dove eqlb^m. ✓

and to show that the pure B eqlb^m is stable too.

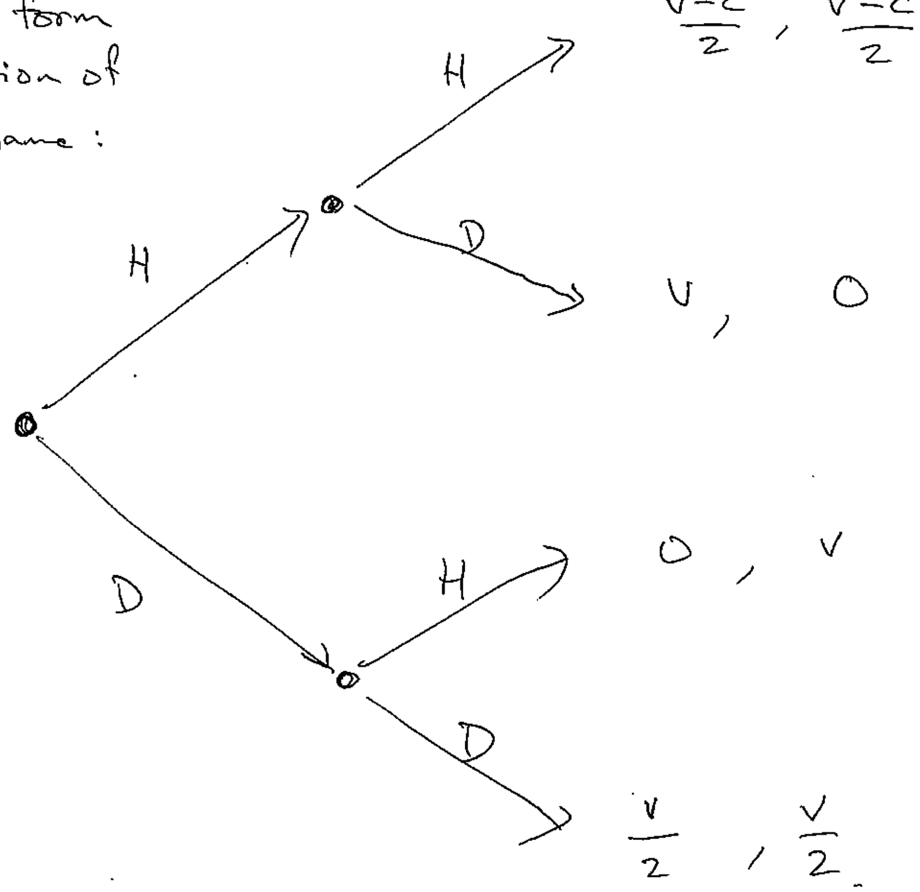
∴ pure B is an ESS.

(in fact it's the only truly stable eqlb^m).

Sequential Play

payoffs:

Called "extensive" form" representation of the H-D game :



1st player chooses.

2nd chooses.

$\boxed{\text{Backwards induction}}$

efficient way to solve (find ESS) in even complex games